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THE GENERALIZED LAGRANGIAN METHOD WITH FLUX LIMITERS

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ABSTRACT

The purpose of this paper is twofold. Firstly, to present a detailed account of the generalized Lagrangian formulation of Hui and Zhao, in which the stream function ξ and Lagrangian distance λ, are used as independent variables, and secondly to assess and compare the performance of various flux limiters in this formulation with their corresponding performance in the Eulerian formulation.

The generalized Lagrangian formulation is obtained by a transformation from the cartesian co-ordinates (x, y) to the Lagrangian co-ordinates (λ, ξ). In this manner, the number of independent variables for steady, 3-D flow is reduced from four to three, placing this formulation on the same footing as the Eulerian formulation even for steady flows (as opposed to the conventional Lagrangian formulation which apparently still requires four independent variables even for steady flows). The generalized Lagrangian formulation with the Godunov scheme (using flux limiters) appears to have distinct advantages over the corresponding Eulerian formulation, particularly with respect to accuracy. Furthermore, the method requires no grid generation.

KEY WORDS Lagrangian method Godunov scheme

INTRODUCTION

The numerical simulation and prediction of inviscid compressible flows is an area of increasing theoretical and practical importance. Based on fundamental work of Godunov[1], a number of upwind, high resolution methods for solving the 1-D unsteady Euler equations have been developed. These and other approaches have been extensively compared and evaluated in Woodward and Collella[2], especially for 2-D flows with strong shocks.

The Euler equations constitute a first order non-linear hyperbolic system of partial differential equations (pde's). Thus, even if the initial data and boundary conditions are sufficiently smooth, discontinuities will always occur in finite time. Physically, this corresponds to the formation of shock waves and slip lines. So, any discretization of the pde's must cope with discontinuities and a good computational method should capture the shock waves and sliplines accurately and correctly.

There are two basic approaches for specifying fluid motion, namely—the Eulerian and Lagrangian formulations. Nearly all existing methods for computing steady flows are based on the Eulerian formulation. This generally resolves slipline discontinuities poorly and always requires the generation of a computational grid to fit the given body shape (which itself is invariably time consuming). Over the past decade Hui and co-workers[3,4] have developed a *new*

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Lagrangian method which uses a Lagrangian time τ and a stream function ξ as independent variables to compute 2-D steady, supersonic flows. They have demonstrated that it is superior to the Eulerian approach in that it resolves sliplines sharply and the shock resolution improves with increasing Mach number. A further advantage to their method is that it requires no grid generation. However, Hui and Zhao[5] have pointed out that, as it stands, the new Lagrangian approach has the following shortcomings:

1. The flux F (see (24)) is discontinuous across a slipline. Although the discontinuous nature of the flux reflects the true dynamics of inviscid flow it can cause problems at the discrete level, and it is thus always desirable to ensure that the numerical fluxes are everywhere continuous.
2. The system is not fully hyperbolic in the sense of Whitham[6]—i.e. although there are six real eigenvalues, there are only five linearly independent associated eigenvectors. This does not appear to affect the numerical method based on the new Lagrangian formulation (see Loh and Hui[4]). However, on a theoretical basis, it is desirable to transform the system to make it fully hyperbolic so that many upwind schemes based on flux splitting can be applied.
3. It is difficult to apply the new Lagrangian formulation to solve subsonic flow problems—since although the boundary corresponds to a streamline, it is difficult to prescribe the boundary condition in terms of τ since the mappings relating the Eulerian and new Lagrangian formulations are unknown in advance. In purely supersonic computations Loh and Hui[4] circumvent this difficulty by using a marching scheme in τ and determining the mappings step by step while marching. More recently, based on experience with the new Lagrangian method, Hui and Zhao[5] have proposed a *generalized Lagrangian* method which uses a Lagrangian distance λ and a stream function ξ as independent variables to compute 2-D steady, supersonic flows.

The purpose of this paper is twofold, firstly to present (in some detail) the generalized Lagrangian formulation and secondly to report on some preliminary numerical computations comparing the performance of various flux limiters when used in this formulation with their corresponding performance in the Eulerian formulation.

GOVERNING EQUATIONS AND THE GENERALIZED LAGRANGIAN FORMULATION

The inviscid flow of a gas is described by the Euler equations expressing conservation of mass, momentum and energy:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_j) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i} = 0 \qquad (i = 1, 2, 3) \tag{2}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}q^2\right)\right] + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left[\rho u_j\left(e + \frac{1}{2}q^2 + \frac{p}{\rho}\right)\right] = 0 \tag{3}$$

Many of the practical problems encountered in the aero-industry can be posed as steady flow problems. For simplicity and ease of exposition we shall describe the generalized Lagrangian formulation for steady 2-D flows. In this case, the governing equations (1)–(3) reduce to:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{4}$$

$$\frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) = 0 \tag{5}$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) = 0 \tag{6}$$

$$\frac{\partial}{\partial x}(\rho uH) + \frac{\partial}{\partial y}(\rho vH) = 0 \tag{7}$$

where u, v are the x, y components of velocity, $H = e + \frac{1}{2}(u^2 + v^2) + p/\rho$ is the specific total enthalpy and e is the specific internal energy. To derive the generalized Lagrangian formulation we integrate equations (4)–(7) over an arbitrary domain Ω, then applying Gauss' divergence theorem, we obtain:

$$\int_{\partial\Omega} \rho v\,\mathrm{d}x - \rho u\,\mathrm{d}y = 0 \tag{8}$$

$$\int_{\partial\Omega} \rho uv\,\mathrm{d}x - (\rho u^2 + p)\,\mathrm{d}y = 0 \tag{9}$$

$$\int_{\partial\Omega} (\rho v^2 + p) - \rho u\,\mathrm{d}y = 0 \tag{10}$$

$$\int_{\partial\Omega} \rho vH\,\mathrm{d}x - \rho uH\,\mathrm{d}y = 0 \tag{11}$$

We now make a transformation of the independent variables from (x, y) to (λ, ξ):

$$\mathrm{d}x = \frac{u}{q}\,\mathrm{d}\lambda + U\,\mathrm{d}\xi \tag{12}$$

$$\mathrm{d}y = \frac{v}{q}\,\mathrm{d}\lambda + V\,\mathrm{d}\xi \tag{13}$$

where $q = (u^2 + v^2)^{1/2}$.

Notice that $\mathrm{d}\xi = 0 \Leftrightarrow \frac{\mathrm{d}x}{v} = \frac{\mathrm{d}y}{u}$ (i.e. ξ is stream function) and that along a streamline (ξ = constant) we have from equations (12) and (13):

$$\mathrm{d}x = \frac{u}{q}\,\mathrm{d}\lambda, \qquad \mathrm{d}y = \frac{v}{q}\,\mathrm{d}\lambda \qquad \text{and} \qquad \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}\lambda^2$$

substituting (12) and (13) into equations (8)–(11) we obtain:

$$\int_{\partial\Omega} K\,\mathrm{d}\xi = 0 \tag{14}$$

$$\int_{\partial\Omega} (Ku + pV)\,\mathrm{d}\xi + \frac{pv}{q}\,\mathrm{d}\lambda = 0 \tag{15}$$

$$\int_{\partial\Omega} (Kv - pU)\,\mathrm{d}\xi - \frac{pu}{q}\,\mathrm{d}\lambda = 0 \tag{16}$$

$$\int_{\partial\Omega} KH\,\mathrm{d}\xi = 0 \tag{17}$$

where $K = \rho(uV - vU)$.

For smooth flows this is equivalent to:

$$\frac{\partial K}{\partial \lambda} = 0 \tag{18}$$

$$\frac{\partial}{\partial \lambda}(Ku + pV) - \frac{\partial}{\partial \xi}\left(\frac{pv}{q}\right) = 0 \tag{19}$$

$$\frac{\partial}{\partial \lambda}(Kv - pU) + \frac{\partial}{\partial \xi}\left(\frac{pu}{q}\right) = 0 \tag{20}$$

$$\frac{\partial}{\partial \lambda}(KH) = 0 \tag{21}$$

These equations together with the following compatibility conditions (obtained from equations (12) and (13)):

$$\frac{\partial U}{\partial \lambda} - \frac{\partial}{\partial \xi}\left(\frac{u}{q}\right) = 0 \tag{22}$$

$$\frac{\partial V}{\partial \lambda} - \frac{\partial}{\partial \xi}\left(\frac{v}{q}\right) = 0 \tag{23}$$

form the generalized Lagrangian formulation of the 2-D steady Euler equations of compressible flow. The system of equations (18)–(23) can be written in a more compact form as:

$$\frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = 0 \tag{24}$$

where

$$\mathbf{E} = \begin{pmatrix} K^{z}Ku + pV \\ Kv - pU \\ KH \\ U \\ V \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ -p\sin\theta \\ p\cos\theta \\ 0 \\ -\cos\theta \\ -\sin\theta \end{pmatrix}$$

TEST PROBLEMS

As a preliminary comparison of the performance of various flux limiters in the generalized Lagrangian and Eulerian formulations, we consider two test problems. The first problem considered is a Riemann problem of 1-D unsteady gas flow in a shock tube with the following initial data:

$$\begin{cases} Q_L: & p = 1 \quad \rho = 1, \quad u = 0 \quad x < 0 \\ Q_R: & p = 0.1, \quad \rho = 0.125, \quad u = 0 \quad x \geqslant 0 \end{cases}$$

for $\gamma = 1.4$.

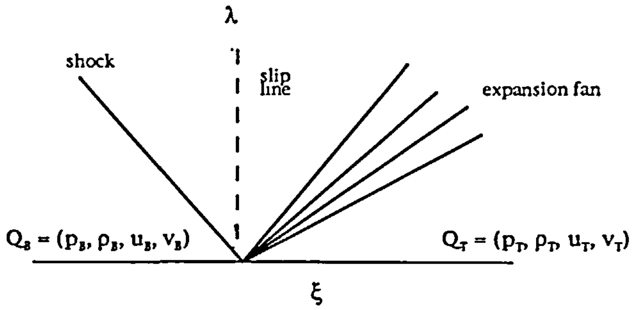

Figure 1

Analogous to the Riemann problem in 1-D unsteady flow, the Riemann problem in 2-D steady supersonic flow is the initial value problem with constant data:

$$\mathbf{Q} = \begin{cases} \mathbf{Q}_T & \xi > 0 \\ \mathbf{Q}_B & \xi < 0 \end{cases}$$

as initial conditions at $\lambda = 0$.

The solution of the Riemann problem is self-similar in the variables ξ/λ and consists of three elementary waves: oblique shock waves, Prandtl–Meyer expansions and slip lines (see *Figure 1*).

Suppose $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the states across one of the elementary waves, then there are three cases: Firstly, suppose the wave is a slip-line, then:

$$p_2 = p_1 \equiv p^*, \qquad \theta_2 = \tan^{-1}\left(\frac{v_2}{u_2}\right) = \theta_1 = \tan^{-1}\left(\frac{v_1}{u_1}\right) \equiv \theta^*$$

but there may be an abrupt change in the density and velocity components. Secondly, suppose the wave is an oblique shock (with $P_2 > p_1$), then from the Rankine–Hugoniot oblique shock relations we obtain for the flow deflection angle:

$$\Delta\theta = \pm\tan^{-1}\left\{\frac{\alpha - 1}{\gamma M_1^2 - \alpha + 1}\left(\frac{2\gamma M_1^2}{(\gamma+1)\alpha + (\gamma-1)} - 1\right)^{1/2}\right\}$$

and

$$\rho_2 = \rho_1 \frac{(\gamma+1)\alpha + \gamma - 1}{(\gamma-1)\alpha + \gamma + 1}$$

$$M_2 = \left[M_1^2 \frac{((\gamma+1)\alpha + \gamma - 1) - 2(\alpha^2 - 1)}{\alpha((\gamma-1)\alpha + (\gamma+1))}\right]^{1/2}$$

where $\alpha = p_2/p_1$.

Finally suppose the wave is an expansion fan (here $p_2 < p_1$), then:

$$M_2 = \left\{\frac{2}{\gamma-1}\left(\frac{1 + ((\gamma-1)/2)M_1^2}{\alpha^{(\gamma-1)/\gamma}} - 1\right)\right\}^{1/2}$$

and

$$\rho_2 = \rho_1 \alpha^{1/\gamma}$$

The flow deflection angle is:

$$\Delta\theta = [\nu(M_2) - \nu(M_1)]$$

where,

$$\nu(M) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan\left(\left[\left(\frac{\gamma-1}{\gamma+1}\right)(M^2-1)\right]^{1/2}\right) - \tan^{-1}(\sqrt{M^2-1})$$

is the Prandtl–Meyer function.

Through any state $\mathbf{Q}_1$, there will be a family of compression states ($\alpha \geqslant 1$) and expansion states ($\alpha < 1$) connecting to $\mathbf{Q}_1$, where $\alpha = p/p_1$. Analogous to 1-D unsteady flow, these two families of curves have second-order contact at $\mathbf{Q}_1$ and can be considered to be one single family (see Reference 7). This forms the basis of the solution procedure used in solving the Riemann problem and we refer the reader to Loh and Hui[4] and Kachura[8] for details of the algorithm. Thus the second test problem considered is the Riemann problem for 2-D steady supersonic flow with initial data:

$$\begin{cases} \mathbf{Q}_T: & p = 0.25, \quad \rho = 0.5, \quad M = 4.0, \quad \theta = 0 \\ \mathbf{Q}_B: & p = 1.0, \quad \rho = 1.0 \quad M = 2.4, \quad \theta = 0 \end{cases}$$

for $\gamma = 1.4$.

FLUX LIMITERS AND NUMERICAL COMPUTATIONS

In this section we carry out a preliminary comparison of various flux limiters used in the solution procedure (see Sweby[9] for details) which are applied to the Eulerian and generalized Lagrangian formulations of the Euler equations of gas dynamics. The test problems considered are the two Riemann problems presented in the previous section. For each test case, the error E between the exact and computed solutions is calculated using the l_1, l_∞ and l_2 norms.

For the first test problem of 1-D unsteady gas flow in a shock tube, *Tables 1* and *2* provide a comparison of the various flux limiters when applied to the Eulerian and generalized Lagrangian formulation respectively.

Overall Roe's superbee flux limiter appears to give the best results both in the Eulerian and generalized Lagrangian formulations. The gain in accuracy is obtained through a sharper

Table 1 Riemann problem for 1-D unsteady flow (Eulerian formulation)

Flux limiter	E_∞	E_1	E_2
Roe's Transfer function	0.560	1.911	0.624
Roe's Superbee	0.518	1.218	0.537
van Leer	0.541	1.531	0.577
van Albada	0.552	1.671	0.599
Chakravarthy–Osher	0.581	1.824	0.631

Table 2 Riemann problem for 1-D unsteady flow (generalized Lagrangian formulation)

Flux limiter	E_∞	E_1	E_2
Roe's Transfer function	0.370	1.439	0.454
Roe's Superbee	0.375	0.884	0.396
van Leer	0.378	1.152	0.424
van Albada	0.369	1.259	0.430
Chakravarthy–Osher	0.395	1.281	0.450

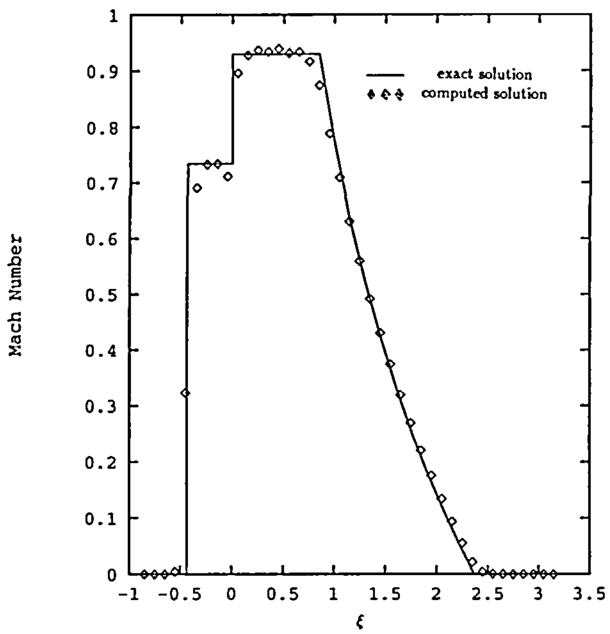

Figure 2

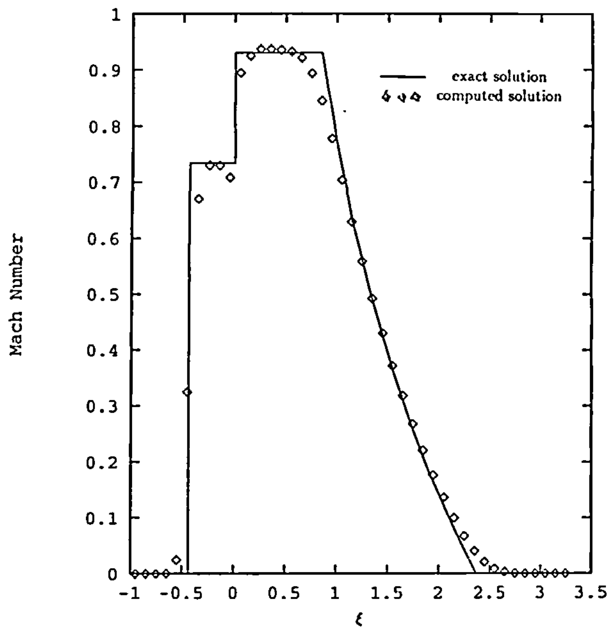

Figure 3

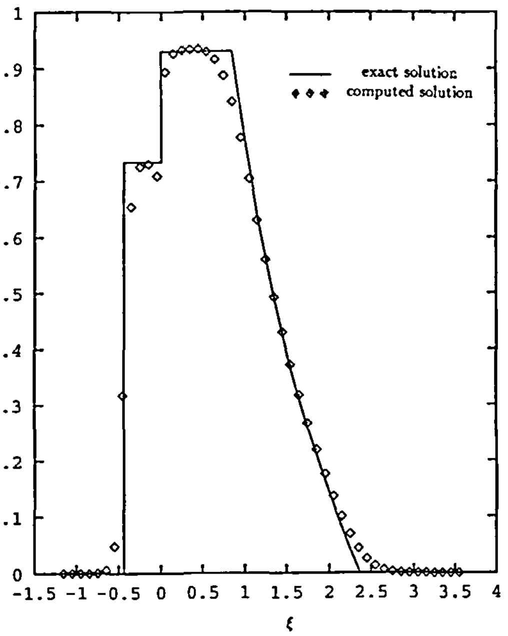

Figure 4

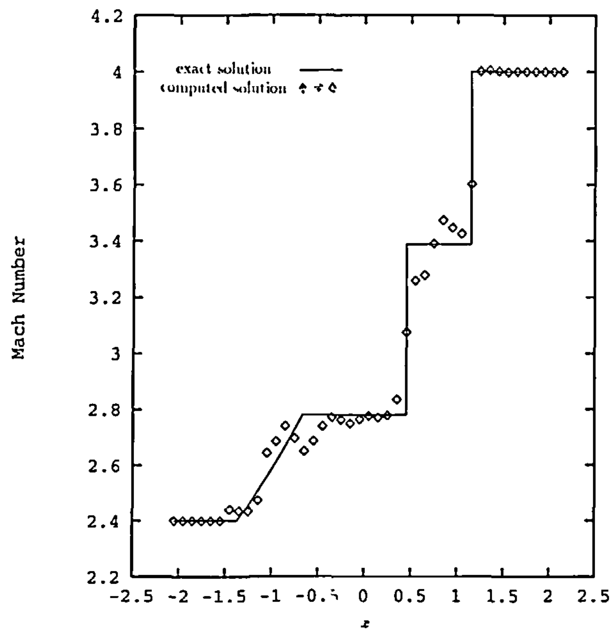

Figure 5

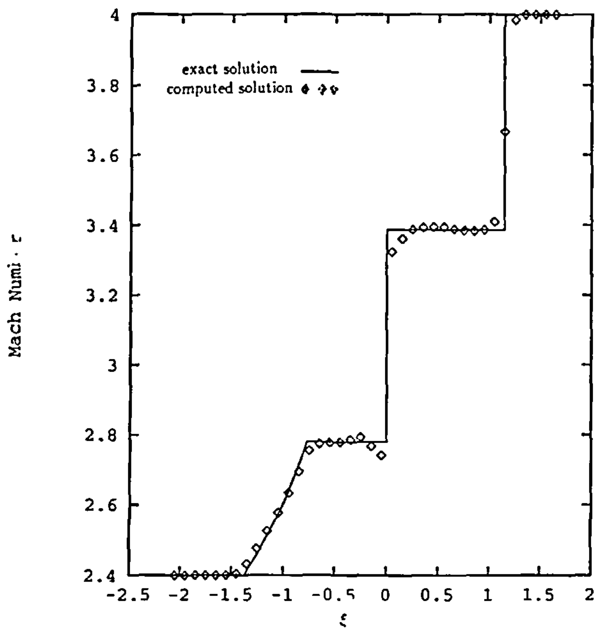

Figure 6

Table 4 Riemann problem for 2-D steady supersonic flow (generalized Lagrangian formulation)

Flux limiter	E_{∞}	E_1	E_2
Roe's Transfer function	0.377	1.267	0.440
Roe's Superbee	0.396	0.852	0.417
van Leer	0.387	1.003	0.423
van Albada	0.379	1.130	0.427
Chakravarthy-Osher	0.399	1.165	0.447

Table 3 Riemann problem for 2-D steady supersonic flow (Eulerian formulation)

Flux limiter	E_{∞}	E_1	E_2
Roe's Transfer function	0.506	2.293	0.699
Roe's Superbee	0.521	1.872	0.655
van Leer	0.801	2.746	0.947
van Albada	0.489	2.011	0.649
Chakravarthy-Osher	0.547	2.185	0.713

resolution of the expansion fan. Accuracy across the shock and slip line is comparable to that for the van Leer or van Albada limiters (see *Figures 2, 3, 4*).

For the second test problem of 2-D steady, supersonic flow, the performance of the various flux limiters is shown in *Tables 3* and *4*.

Although in this case, no single flux limiter performs significantly better than the others, we can conclude from the tables that, for each flux limiter tested, the results using the generalized Lagrangian formulation prove superior to those using the Eulerian formulation. *Figures 5* and

6 show the performance of Roe's superbee limiter in the Eulerian and generalized Lagrangian formulations. The performance of the flux limiter in the Eulerian formulation, at this high Mach number, has deteriorated significantly, while its performance in the generalized Lagrangian formulation appears to be qualitatively accurate.

CONCLUSION

In conclusion, the generalized Lagrangian method using flux limiters appears to have the following advantages over the corresponding Eulerian method: sliplines are resolved more sharply, shock resolution improves considerably with Mach number and no grid generation is required. Further, the generalized Lagrangian method, while retaining all the merits of the new Lagrangian method, overcomes some of the existing deficiencies of the latter (see Hui and Zhao[5]), and apart from producing more accurate and reliable results than the Eulerian formulation is actually easier to program—since the boundary between two adjacent cells always coincide with the slipline, the flux there required by the Godunov scheme is easier to compute; in contrast, the cell may lie in any of five regions separating the elementary waves, in the Eulerian formulation, resulting in more complicated programming.

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